# Recent Developments in Rotary-Wing Aeroelasticity

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## Nomenclature

$[A(\psi)]$ = periodic matrix
$[A]$ = constant matrix, symbolic
$a$ = lift curve slope, two-dimensional
$\bar{a}$ = offset between elastic axis and midchord, positive aft, nondimensionalized with respect to $R$
$b$ = semichord, nondimensionalized with respect to $R$
$B_1, B_2$ = blade cross-sectional integrals
$[B]$ = constant matrix, symbolic
$C_{DP}$ = drag coefficient due to equivalent flat-plate area of the helicopter
$C_w$ = weight coefficient approximately equal to $C_T$
$C_T$ $= T/\rho_A(\pi R^2\Omega^2 R^2)$ = thrust coefficient
$\{C\}$ = constant column matrix
$C_{d0}$ = profile drag coefficient
$C(k)$ = Theodorsen's lift deficiency function
$C'(k,h,m)$ = Loewy's modified lift deficiency function
$[D]$ = constant matrix, symbolic
$E_{c1}, E_{c2}, E_{c3}$ = terms associated with elastic coupling
$(EI)_y, (EI_z)$ = stiffness for flapwise and inplane bending, respectively
$E$ = Young's modulus
$[F]$ = constant matrix, symbolic
$f_1$ = generalized coordinate, first torsional mode
$f_1^0$ = static value of $f_1$ in hover
$\Delta f_1$ = perturbation of $f_1$ about $f_1^0$
$g_1$ = generalized coordinate, first normal flapping mode
$\Delta g_1$ = perturbation in $g_1$ about $\bar{g}_1$
$g_1^0$ = static value of $g_1$ in hover, or constant part of $\bar{g}_1$
$\bar{g}_1$ = linear time dependent equilibrium value of first normal flapping mode
$GJ$ = torsional stiffness
$g_{1c}, g_{1s}$ = cyclic parts of $\bar{g}_1$
$h$ = nondimensional wake spacing
$h_1$ = generalized coordinate, first normal inplane mode
$\Delta h_1$ = perturbation in $h_1$ about $\bar{h}_1$
$h_1^0$ = constant part of $\bar{h}_1$, or static value in hover
$h_{1s}, h_{1c}$ = cyclic parts of $\bar{h}_1$
$\bar{h}_1$ = linear time dependent equilibrium value of first normal lead lag mode
$[I]$ = unit matrix
$i$ $= \sqrt{-1}$
$\mathbf{i,j,k}$ = unit vectors in $x$, $y$, and $z$ direction (Fig. 1)
$\mathbf{I_2,J_2,K_2}$, $\mathbf{I_3,J_3,K_3}$ = unit vectors defining deformed blade geometry (shown in Fig. 2); $\mathbf{J_2}$ is parallel to hub plane, $\mathbf{I_2}$ and $\mathbf{I_3}$ are tangential to the deformed blade elastic axis.
$k$ = reduced frequency
$k_A$ = polar radius of gyration of cross-sectional area effective in carrying tensile stresses about the elastic axis ($\bar{k}_A = k_A/l$)
$k_0$ = polar radius of gyration of cross-sectional mass about its center of gravity ($\bar{k}_0 = k_0/l$)
$k_\phi$ = root torsional spring constant, control system stiffness
$l$ = length of blade capable of elastic deformation
$m = \omega/\Omega$ = frequency ratio
$M$ = Mach number at radial station $r = x_0 + e_1$
$P_x, P_y, P_z$ = resultant total loadings per unit length in the $x$, $y$, and $z$ directions, respectively; subscript $I$ denotes inertia
$q_x, q_y, q_z$ = distributed external loading torques in the $x$, $y$, and $z$ directions, respectively
$\{q_0\}$ = vector of variables defining time-dependent equilibrium position of the blade
$[P(\psi)]$ = periodic matrix in Floquet-Liapunov Theorem
$R$ = blade radius
$[R]$ = constant matrix used in Floquet-Liapunov Theorem
$[S]$ = matrix used in calculating equilibrium position, symbolic
$T$ = centrifugal tension in the blade, also common nondimensional period used in the Floquet theory, also thrust in trim procedure
$u, v, w$ = $x$, $y$, and $z$ displacements of a point on the elastic axis of the blade

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$v_e, v_{e0}$ = elastic part of the displacement of a point on the elastic axis of the blade parallel to the hub plane (see Fig. 1); subscript 0 denotes equilibrium position

$V$ = velocity of forward flight of the whole rotor

$w_e, w_{e0}$ = elastic part of the displacement of a point on the elastic axis of the blade, in the $\boldsymbol{K}_2$ direction (Fig. 2); subscript 0 denotes equilibrium position

$x, y, z$ = rotating orthogonal coordinate system

$x_0 = x - e_l$ = running spanwise coordinate for part of the blade free to deflect elastically, $x_l$ - same, dummy variable

$x_I, (\bar{x}_I = x_I / l)$ = blade cross-sectional mass center of gravity offset from the elastic axis (Fig. 1b)

$x_A, (\bar{x}_A = x_A / l)$ = blade cross-sectional aerodynamic center offset from elastic axis, shown in Fig. 1b, positive for A.C. before E.A.

$\{y\}$ = state variable column matrix

$\beta_P$ = precone, inclination of feathering axis with respect to the hub plane measured in a vertical plane

$\gamma$ = lock number ($\gamma = 2\rho_A bR^5 a / I_b$) for normal flow

$\gamma_I$ = first inplane bending mode

$\epsilon_D$ = symbolic quantity having the same order of magnitude as the slopes in the flap and lag directions.

$\bar{\zeta}_k$ = real part of the $k$th characteristic exponent

$\zeta_k$ = real part of $k$th eigenvalue

$\eta_I$ = first flapwise bending mode

$\eta_{SFI}, \eta_{SLI}$ = viscous structural damping coefficients, in percent of critical damping, for first flap and lag mode, respectively

$\theta_0$ = collective pitch angle

$\theta_B$ = built-in twist

$\theta_G$ = total geometric pitch angle

$\theta_t$ = time-dependent part of geometric pitch angle

$\theta_{Ic}, \theta_{Is}$ = cyclic pitch components

$\theta_c$ = critical value of collective pitch at which linearized coupled flap-lag system becomes unstable in hover

$\lambda$ = symbolic eigenvalue; also inflow ratio, induced velocity over disk, positive down, nondimensionalized with respect to $\Omega R$

$\lambda_k$ = eigenvalues of $[R]$, characteristic exponents

$\mu$ = advance ratio

$\sigma$ = blade solidity ratio

$[\Phi(\psi, \psi_0)]$ = state transition matrix at $\psi$, for initial conditions given at $\psi_0$

$\phi$ = total elastic torsional deformation

$\phi_I$ = first torsional root-coupled mode

$\psi$ = azimuth angle of blade ($\psi = \Omega t$) measured from straight aft position

$\omega_k$ = imaginary part of $k$th eigenvalue

$\bar{\omega}_k$ = imaginary part of $k$th characteristic exponent

$\bar{\omega}_{FI}, \bar{\omega}_{LI}, \bar{\omega}_{TI}$ = natural frequency of first flap, lead-lag, and torsional frequency, respectively, nondimensionalized with respect to $\Omega$

$\bar{\omega}_{\phi I}$ = same as $\bar{\omega}_{TI}$

$\omega$ = flutter frequency

$\Omega$ = speed of rotation

*Special Symbols*

$(\ )$ = $d/d\psi$

$[\ ]$ = square matrix, $[\ ]^{-1}$ inverse, $[\ ]^T$ transpose

$\{\ \}$ = column matrix

## I. Introduction

AEROELASTICITY deals with the behavior of an elastic system in an airstream wherein there is a significant reciprocal interaction or feedback between deformation and flow. Although dramatic instabilities may often be featured, it may be stressed at the outset that control of subcritical response behavior of the system is as important as removal of such critical conditions from the design or flight envelopes. Thus, the main problems of aeroelasticity are determination of both the response and the stability problems. Furthermore, aeroelasticity deals with the interaction of internal and external sources of energy; thus it includes servo-aeroelasticity and significantly interacts with the area of flight dynamics.

Dynamic stability and response problems associated with rotary-wing aircraft represent some of the most complex problems in the area of aeroelasticity. Due to the complicated nature of these aeroelastic problems, it is not surprising to find that this part of aeroelasticity is considerably less developed and advanced than its fixed-wing counterpart. A considerable amount of significant research in this area has been done primarily during the past 25 years. However it is interesting to note that classical texts on aeroelasticity[1,2] do not even mention this important and challenging subject area. Only recently, in an extensive treatise on aeroelasticity by Forsching,[3] are aeroelastic problems associated with rotors identified as a new and significant part of aeroelasticity; however, their treatment in this text is quite superficial, which is understandable in light of the large number of topics covered in this book.

A positive sign of the growing awareness of the importance of rotary-wing aeroelasticity is evident from Garrick's[4] recent review of aeroelasticity, which is primarily devoted to some relatively complex problems of fixed-wing aeroelasticity such as active control of aeroelastic response including the control of flutter, unsteady aerodynamics of arbitrary configurations, and problems of gust response. In this review, where aeroelastic problems of rotorcraft are only very briefly mentioned, Garrick identifies the aeroelastic problems of helicopters as one of the remaining fertile areas for advanced work in aeroelasticity, indicating that many fundamental problems in this area still remain to be considered.

One of the first significant reviews of rotary-wing V/STOL dynamic and aeroelastic problems was undertaken by Loewy[5] who reviewed the literature up to the end of 1968 with considerable detail and great physical insight. In this review, a wide range of topics were considered such as: static and dynamic classical coupled flap-pitch problems,[6] flap-lag flutter, pitch-lag flutter, propeller whirl and prop-rotor whirl flutter, mechanical instability (ground resonance), coupled airframe/rotor instabilities in flight (air resonance), problems associated with the effect of forward flight and periodic coefficients, stall flutter, and some special problems.

A somewhat more restricted review, intended primarily to bring up to date the chapter dealing with helicopter blade flutter in the Agard Manual on Aeroelasticity has been written by Ham[7]; however, it was limited to the discussion of classical coupled flap-torsion flutter,[6] flap-lag flutter without the structural elastic coupling effect, and stall flutter.

Another review of rotary-wing aeroelastic problems has been written by Dat.[8] This review, which is somewhat limited in scope because it has been written primarily for instructional purposes, contains among other topics, an excellent treatment of the unsteady aerodynamic problem for rotary-wing aircraft and a review of aeroelastic response and vibration problems associated with forward flight.

The research-and-development work done on the flight dynamics problems of hingeless rotorcraft in the NATO countries up to 1973 has been the subject of an excellent and comprehensive review report written by Hohenemser.[9] Due to the strong interaction between blade elastic deformations and flight dynamics, some aeroelastic problems are included in Ref. 9, since they are considered to be a part of the broader flight dynamics problem. Typical of these are the description of coupled rotor/fuselage and rotor/fuselage control system aeroelastic problems.

Finally it should be noted that Soviet research on rotary-wing aeroelasticity is described with considerable detail in Mil's translated two-volume treatise,[10,11] and research on steady and unsteady rotary-wing aerodynamics is described in a book by Baskin et al.,[12] which has been translated recently.

During the past 8-year period since Loewy's[5] review, significant advances in rotary-wing aeroelasticity have occurred. The objective of the present paper is to review, with considerable detail the most important aspects of this research. Although the author's own work will be used to illustrate some of the points made, a considerable effort was made to present recent research in a balanced and objective manner. The most important development in rotary-wing aeroelasticity during the past 8-year period has been the acceptance of the fact, by industry, researchers, and the academic community, that rotary-wing aeroelasticity is *inherently nonlinear*. Thus the correct treatment of a wide class of problems in this area requires a *consistent* development of a mathematical model, for the particular aeroelastic problem being considered, which results in a system of nonlinear equations of motion.

This review will make an attempt to cover the following topics:

1) Recent developments in the aeroelastic modeling of the coupled flap-lag-torsional problem in hover.

2) Review of unsteady aerodynamic theories applicable to rotary-wing aeroelasticity and their incorporation in aeroelastic analyses dealing with the coupled flap-lag-torsional aeroelastic problem in hover.

3) Review of the coupled flap-lag and coupled flap-lag-torsional aeroelastic problem of rotor blades in forward flight. The importance of trim and nonlinear terms on blade aeroelastic stability for this case is discussed. Furthermore, efficient numerical methods for treatment of equations with periodic coefficients are also reviewed.

4) The complete rotor and coupled rotor fuselage aeroelastic problem is reviewed with considerable detail. Both hingeless and teetering rotors are considered. Tail rotor aeroelastic problems are also briefly reviewed.

This review attempts to present the state-of-the-art of rotary-wing aeroelasticity from a unified viewpoint. It should be noted, however, that relatively more space is devoted to hingeless rotor configurations than to teetering or articulated rotor configurations. This is mainly due to the fact that a considerable part of recent research has dealt with hingeless rotors.

## II. Mathematical Modeling of the Coupled Flap-Lag-Torsional Aeroelastic Problem in Hover

### A. Introduction

The coupled flap-lag-torsional aeroelastic problem in hover is the fundamental problem in rotary-wing aeroelasticity. Coupled flap-pitch,[6] coupled pitch-lag,[13] and more recently the coupled flap-lag[14-16] problem have all received attention in rotary-wing aeroelasticity. However, it is only the more recent research which has shown that, for most cases when the first torsional frequency of the blade is below $\bar{\omega}_{\phi l} < 9$, the coupled flap-lag-torsional problem has to be considered in its entirety in order to obtain results that have practical value. It is also important to realize that whereas flap-pitch and pitch-lag instabilities were initially obtained with linear formulations of the aeroelastic problem, the correct treatment of the flap-lag type of instability requires the derivation of nonlinear equations of motion. Correct flap-lag stability boundaries can be obtained only from a properly linearized version of these equations. Thus it is clear that in order to have a mathematical model representing the coupled flap-lag-torsional motion of a blade that contains the coupled flap-pitch, pitch-lag, and flap-lag problems as subsets it is necessary to derive a set of nonlinear coupled flap-lag-torsional equations. In this section, the most important aspects related to the derivation of coupled flap-lag-torsional equations of motion and their solution will be reviewed.

### B. Formulation of Coupled Flap-Lag-Torsional Equations of Blade Motion

The coupled flap-lag-torsional equations for a single blade are the basic building block from which the more complicated aeroelastic problems can be developed. For hingeless rotor blades, where the blades are cantilevered to the hub, recent research, described in this section, has shown that nonlinear equations of blade motion have to be derived. The nonlinear terms in these equations are due to the inclusion of moderately large deflections in the elastic, inertial, and aerodynamic operators of this aeroelastic problem. Physically these moderately large deflections correspond to a situation where one has small elastic strains combined with finite rotations (large slopes). It has also been shown that moderately large deflections can also be important in teetering and articulated rotor blades.

During the past few years a number of equations of motion for the coupled flap-lag-torsional motion of rotor blades have been derived by a number of authors.[15,17-26] A number of these derivations have been directed at the hingeless rotor aeroelastic problem,[15,17,19-21,23-26] whereas others were more general in nature and were capable of simulating hingeless, articulated, and teetering blade configurations by appropriate modification of the blade boundary conditions at the root.[18,22] However it should be noted that by proper adjustment of boundary conditions and use of appropriate mode shapes the equations derived for hingeless rotor blades can also be applied to the articulated rotor blade problem. The teetering rotor represents a special situation that has to be handled in a different manner as will be shown later in this paper.[54,55]

In order to be able to review and compare the various equations available in a systematic manner it is important to describe briefly their salient features, the assumptions upon which they are based, and the methods used in their derivation. The geometry of a typical hingeless rotor blade having flap, lag, and torsional degrees of freedom is shown in Figs. 1 and 2. A slightly more general configuration having both sweep and droop is shown in Fig. 3. A fundamental set of equations for blades having no droop, sweep, and precone has been presented by Houbolt and Brooks[17] where equations of equilibrium for the coupled bending and torsion of a pretwisted nonuniform blade has been derived. All three, flap, lag, and torsional degrees of freedom were taken into account, the final equations obtained were intended to represent only the *linear problem*. Furthermore, the aerodynamic loading terms were left in a general unspecified form.

Following this work other researchers presented derivations of equations that included additional nonlinear terms due to moderately large deflections. When nonlinear terms are included in the structural, inertia, and aerodynamic operators of this aeroelastic problem some difficulties are encountered:

1) A considerable number of higher order terms are obtained. In order to neglect the appropriate terms a rational ordering scheme has to be used enabling one to neglect terms in a systematic manner. In such an ordering scheme, all important parameters of the problem are assigned orders of magnitude in terms of typical nondimensional displacement

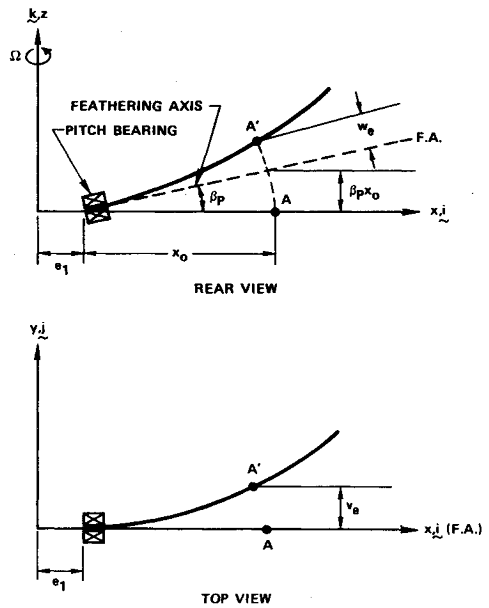


**Fig. 1a Deformed elastic axis for a typical cantilevered preconed rotor blade.**

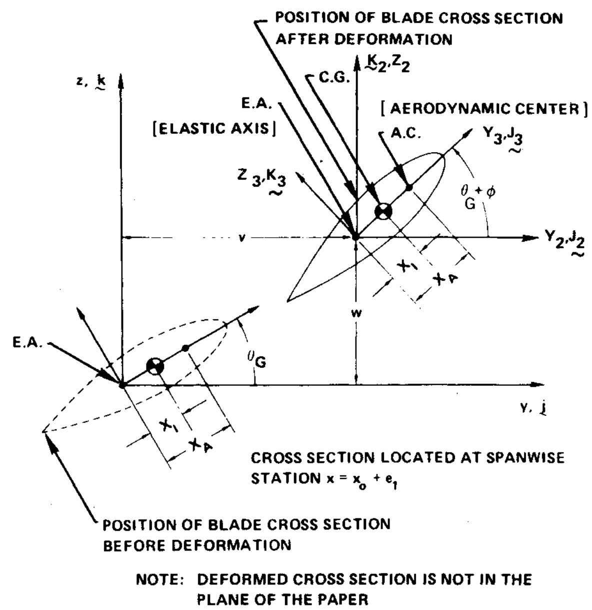


**Fig. 1b Blade model and position of the cross section before and after the deformation.**

quantity $\epsilon_D$, which represents typical blade slopes; thus

$$\frac{v}{R}=\frac{w}{R}=\phi=\lambda=\beta_p=b=\frac{\partial\theta_B}{\partial x_0}=\frac{k_0}{R}=0(\epsilon_D)$$

$$\frac{x_0}{R}=\frac{\partial}{\partial x_0}=\frac{\partial}{\partial\psi}=0(1)$$

$$\theta=0(\epsilon_D^{1/2})$$

$$C_{d0}/a=x_I/R=0(\epsilon_D^2)$$

This ordering scheme is used with the assumption that terms of $0(\epsilon_D^2)$ are usually neglibible when compared to terms of order one. In most cases such ordering schemes are used with a certain amount of flexibility. An alternative approach for neglecting higher order terms in the equations of motions is based upon the concept of expanding the various expressions

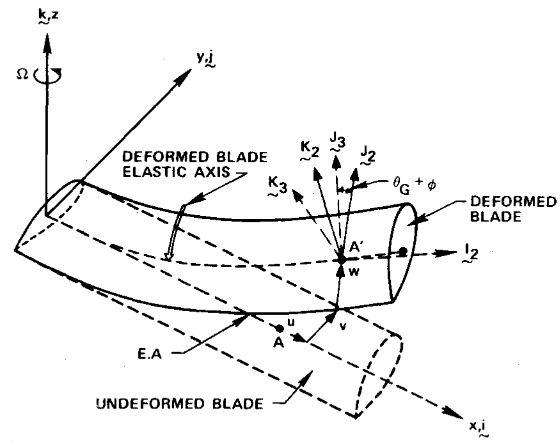


**Fig. 2 Schematic representation of deformed and undeformed blade.**

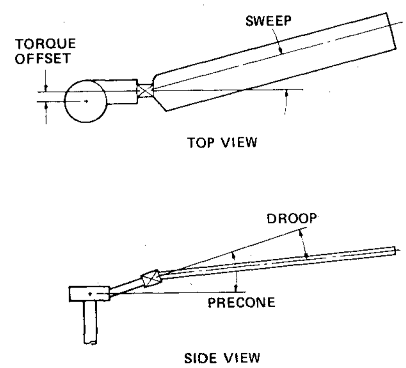


**Fig. 3 General blade configuration with sweep and droop.**

in the equations of motion in a Taylor series in the vicinity of an appropriate equilibrium position. Approximate equations are obtained by neglecting higher order terms in the Taylor series expansion.

2) Care must be exercised in distinguishing between the deformed and undeformed positions of the blade.

With this information the various nonlinear equations available can be reviewed. A detailed set of linearized coupled flap-lag-torsional equations have been derived by Arcidiacono.[18] These equations were derived using the Taylor series method of approximation. They are suitable for both articulated and nonarticulated blades. Fully coupled aerodynamic forcing functions were included based upon quasisteady aerodynamic theory. The differential equations of motion were expanded in terms of the uncoupled vibratory modes of the blade.

Another system of coupled flap-lag-torsional equations of motion, based upon the ordering scheme method of approximation has been derived by Friedmann and Tong.[15] In these equations, the torsional degree of freedom was represented by a root torsional spring, i.e., pitch link flexibility. The aerodynamic loads were modeled using quasisteady aerodynamic theory. Various cross-sectional blade offsets shown in Fig. 1 were included in the derivation. Since their initial derivation, these equations have undergone a continuous process of improvement and modification. A more recent version of these equations[21] includes a modified structural operator similar to Houbolt and Brooks except that it contains moderately large deflections. Furthermore these improved equations are capable of modeling root torsional deformations, distributed torsional deformations, blade precone, and the various cross-sectional offsets shown in Fig. 2.

When droop is included in the formulation of the mathematical model, the coupled flap-lag-torsional equations become much more complicated, because for this case the undeformed elastic axis of the blade moves on a cone in space whenever collective pitch angles or root torsional deformations are introduced. The equations presented in Ref. 21 have been recently generalized by Reyna-Allende[26] to include droop, time-varying pitch angle associated with forward flight and aerodynamic loads that include the effects of forward flight.

A comprehensive set of coupled flap-lag-torsional equations of motion has been presented by Hodges and Dowell,[20] these are an improvement of the equations first presented in Ref. 19. These equations were developed from nonlinear strain displacement relations, using both Hamilton's principle and the Newtonian method. The neglection of the higher order nonlinear terms is based upon the ordering scheme type of approximation. In Ref. 20, the aerodynamic loads have been left in general unspecified form; however, in Ref. 19 quasisteady blade element theory was used for airload computation. The equations of Ref. 20 are limited to the case of distributed torsional representation of blade flexibility and hover. A more recently published version of these equations[24] also contains variable structural coupling and a quasisteady representation of the aerodynamic loads. They apply only for the case of hover.

In a recent preliminary report[25] Hodges has generalized the equations of Ref. 20 to include pitch link flexibility twist, precone, droop, sweep, and torque offset. Quasisteady aerodynamic loads for the equations were also derived. The equations are somewhat limited by the assumption that the cross-sectional elastic axis, center of mass, and tension are coincident at the quarter chord. Thus none of the offsets shown in Fig. 2 are included. Furthermore, the equations are limited to the case of hover.

A general set of coupled flap-lag-torsional equations of motion of a single blade, which is part of a more general analysis aimed at dealing with complete coupled rotor-fuselage aeroelastic problem, has been developed by Johnston and Cassarino.[22] These equations are general and capable of simulating gimbaled, articulated, and hingeless rotor-blade configurations. The equations are obtained using a Lagrangian approach, and are linearized about an appropriate equilibrium position by using the Taylor-series expansion method. The unsteady aerodynamic loads can be included using two-dimensional strip theory based upon the Theodorsen and Loewy type of life-deficiency functions. Tabulated airfoil data, stall, and compressibility can be taken into account. The equations are applicable to both hover and forward flight.

Another advanced and comprehensive set of equations of motion aimed at modeling a composite bearingless rotor blade has been recently developed by Bielawa.[23,27] In these equations the neglection of the higher order nonlinear terms is based upon an ordering scheme type approximation. The equations are derived using a Newtonian approach and can be used for both hover and forward flight. These equations are a good example of the "art" of modeling a complex, composite rotor blade where structural redundancies and nonlinear twist are present.

Finally, it also should be noted that recently an accurate set of nonlinear equations of equilibrium for elastic helicopter blades undergoing moderate deformation has been derived by Rosen and Friedmann.[66]

These equations are based on the assumption of small strains and finite rotations. Only the structural operator is presented in an explicit manner, the inertia and aerodynamic operators are left in a general form. These equations, which are slightly more accurate and consistent than the equations presented in Refs. 21 and 26, are suitable for a variety of rotary-wing aeroelastic applications.

#### C. Solution of the Coupled Flap-Lag-Torsional Problem in Hover and Results

The fundamental difference between the fixed-wing and the rotary-wing aeroelastic problem is the simple fact that the rotary-wing aeroelastic problem is inherently nonlinear. This aspect of rotary-wing aeroelasticity has not been sufficiently emphasized in previous reviews of rotary-wing aeroelasticity which tended to emphasize the effects of rotation and the complicated nature of rotary-wing unsteady aerodynamics. The various equations of motion described in the previous section retain without exception the most important nonlinear terms in the aerodynamic and inertia operators and in most cases also in the structural operators associated with this aeroelastic problem. One may consider this to be evidence that the inherently nonlinear character of rotary-wing aeroelasticity has become generally accepted.

The consequences associated with the nonlinear nature of the rotary-wing aeroelastic problem can be best appreciated by presenting a brief, symbolic, outline of the solution of the coupled flap-lag-torsional problem for a hingeless blade in hover.

The equations of motion for the problem can be taken from Ref. 21 and correspond to the geometry of the problem shown in Figs. 1 and 2. For this example, one can assume without any loss in generality that there is no built-in twist $\theta_B = 0$ and furthermore $EB_1 = EB_2 = k_A = 0$, the partial differential nonlinear equations of equilibrium are given by

$$\frac{\partial^2}{\partial x_0^2}\left\{[(EI)_y + E_{c1}]\frac{\partial^2 w_e}{\partial x_0^2} + E_{c2}\frac{\partial^2 v_e}{\partial x_0^2} + 2E_{c2}\phi\frac{\partial^2 w_e}{\partial x_0} + E_{c3}\phi\frac{\partial^2 v_e}{\partial x_0^2}\right\} - \frac{\partial}{\partial x_0}\left(T\frac{\partial w}{\partial x_0}\right) - \frac{\partial q_y}{\partial x_0} = P_z \tag{1}$$

$$\frac{\partial^2}{\partial x_0^2}\left\{E_{c2}\frac{\partial^2 w_e}{\partial x_0^2} + [(EI)_z - E_{c1}]\frac{\partial^2 v_e}{\partial x_0^2} + E_{c3}\phi\frac{\partial^2 w_e}{\partial x_0^2} - 2\phi E_{c2}\frac{\partial^2 v_e}{\partial x_0^2}\right\} - \frac{\partial}{\partial x_0}\left(T\frac{\partial v}{\partial x_0}\right) + \frac{\partial q_z}{\partial x_0} = P_y \tag{2}$$

$$-\frac{\partial}{\partial x_0}\left(GJ\frac{\partial \phi}{\partial x_0}\right) + \left\{\frac{\partial^2 v_e}{\partial x_0^2}\frac{\partial^2 w_e}{\partial x_0^2}(E_{c3} - E_{c1}) + E_{c2}\left[\left(\frac{\partial^2 w_e}{\partial x_0^2}\right)^2 - \left(\frac{\partial^2 v_e}{\partial x_0}\right)^2\right]\right\} + E_{c3}\left[\left(\frac{\partial^2 w_e}{\partial x_0^2}\right)^2 - \left(\frac{\partial^2 v_e}{\partial x_0^2}\right)^2\right]\phi = q_x + q_y\frac{\partial v}{\partial x_0} + q_z\frac{\partial w}{\partial x_0} \tag{3}$$

where

$$T = \int_{x_0}^{R} P_{x1}\,\mathrm{d}x_0 \tag{4}$$

$$E_{c1} = [(EI)_z - (EI)_y]\sin^2\theta_G \tag{5}$$

$$E_{c2} = [(EI)_z - (EI)_y]\sin\theta_G\cos\theta_G \tag{6}$$

$$E_{c3} = [(EI)_z - (EI)_y]\cos^2\theta_G \tag{7}$$

and the displacement field for a point on the elastic axis of the blade is given by

$$u = -\beta w_e - \frac{x_0}{2}\beta_p^2 - \frac{1}{2}\int_0^{x_0}\left[\left(\frac{\partial w_e}{\partial x_1}\right)^2 + \left(\frac{\partial v_e}{\partial x_1}\right)^2\right]\mathrm{d}x_1 \tag{8}$$

$$v = v_e \tag{9}$$

$$w = w_e + \beta_p x_0 \tag{10}$$

The quantities $p_x$, $p_y$, and $p_z$ represent distributed loads along the elastic axis of the blade in the $x$, $y$, and $z$ directions, respectively; they contain inertia, aerodynamic, and structural damping loads,[21] whereas $q_x$, $q_y$, and $q_z$ represent the corresponding torques. Complete expressions for these loads are given in Ref. 21 upon which this analysis is based. It is important to remember that the loads $p_y$, $p_z$,..., $q_x$,... etc. will be in general nonlinear functions of the elastic degrees of freedom and their derivatives, i.e. $p_y = p_y$ ($v_e$, $w_e$, $\phi$, $v_e'$, $w_e'$, $\phi'$...)...etc.

The terms denoted by $E_{c1}$, $E_{c2}$, and $E_{c3}$ in these equations represent the elastic coupling terms, whereas the second term in the brackets in Eq. (3) represents the so called torsion-flap-lag coupling or Mil-type term.[10,28,29]

The system of general, coupled, partial differential equations† of motion presented above is transformed into a system of ordinary nonlinear differential equations by using Galerkin's method to eliminate the spatial variable. In this process, the elastic degrees of freedom in the problem are represented by the uncoupled free vibration modes of a rotating blade.

For simplicity, in the derivation presented below, one elastic mode is used to represent each elastic degree of freedom. However the procedure can be easily generalized to an arbitrary number of modes. Using this single mode representation one has

$$w_e = l\eta_1(\bar{x}_0)g_1(\psi) \qquad v_e = -l\gamma_1(\bar{x}_0)h_1(\psi)$$
$$\phi = \phi_1(\bar{x}_0)f_1(\psi) \tag{11}$$

The process of linearization consists of expressing the time dependence of the generalized coordinates of the elastic displacement field as the sum of a steady-state value about which time-dependent perturbations occur.

$$g_1(\psi) = g_1^0 + \Delta g_1(\psi) \qquad h_1(\psi) = h_1^0 + \Delta h_1(\psi)$$
$$f_1(\psi) = f_1^0 + \Delta f_1(\psi) \tag{12}$$

Equations (12) are substituted into the ordinary nonlinear coupled equations of motion and terms containing squares of the perturbation quantities $\Delta g_1$, $\Delta h_1$, and $\Delta f_1$ are neglected. The static equilibrium position is obtained from

$$[S]\begin{Bmatrix} g_1^0 \\ h_1^0 \\ f_1^0 \end{Bmatrix} = \{C\} + \begin{Bmatrix} F_{SN}(g_1^0, h_1^0, f_1^0) \\ L_{SN}(g_1^0, h_1^0, f_1^0) \\ T_{SN}(g_1^0, h_1^0, f_1^0) \end{Bmatrix} \tag{13}$$

where $F_{SN}$, $L_{SN}$, and $T_{SN}$ are lengthy nonlinear expressions of the static equilibrium position associated with the flap, lag, and torsional degrees of freedom, respectively.

The final form of the dynamic equations of equilibrium can be symbolically written as (′ denotes differentiation, $\partial/\partial\psi$)

$$[A]\{q''\} + [B]\{q'\} + [D]\{q\} = 0 \tag{14}$$

where $\{q\}^T = \lfloor \Delta g_1, \Delta h_1, \Delta f_1 \rfloor$ and the matrices $[A]$, $[B]$, $[D]$ are functions of the static equilibrium position only, given by Eq. (13).

In order to transform Eq. (14) into the standard eigenvalue form, Eq. (14) is rewritten in state variable form

$$\{q'\} = \{y_1\} \qquad \{q\} = \{y_2\} \tag{15}$$

this transformation yields

$$\{y'\} = [F]\{y\} \tag{16}$$

where

$$\{y\}^T = \lfloor \{y_1\}^T, \{y_2\}^T \rfloor$$

†With boundary conditions corresponding to a cantilevered blade.

and

$$[F] = \left[\begin{array}{c|c} -[A]^{-1}[B] & -[A]^{-1}[C] \\ \hline [I] & [0] \end{array}\right] \tag{17}$$

Assuming solutions for Eq. (16) in the form of $\{y\} = \{\bar{y}\}e^{\lambda\psi}$ reduces the problem to the standard eigenvalue problem

$$[F]\{y\} = \lambda\{y\} \tag{18}$$

When quasisteady aerodynamics are used, the matrix $[F]$ is real and solution of the stability problem is straightforward, as indicated below.

The exact solution to Eqs. (13) and (18) is obtained by first solving the set of nonlinear algebraic equations represented by Eqs. (13) using a numerical method such as the Newton-Raphson iterative method. Failure of this method to converge usually can be associated with nonlinear coupled flap-lag-torsional divergence or static instability.[15,30]

Using the values of $g_1^0$, $h_1^0$, and $f_1^0$ the eigenvalue problem represented by Eq. (18) can be easily solved. The eigenvalues appear in complex conjugate pairs

$$\lambda_k = \zeta_k \pm i\omega_k \tag{19}$$

The system is stable when $\zeta_k < 0$ and the stability boundary is obtained from $\zeta_k = 0$.

To complete the treatment of this problem, some typical results, which can be obtained from the analysis described above, will be presented. For the sake of clarity first, a few results associated with the coupled flap-lag problem will be given and next the coupled flap-lag-torsional results are presented.

*1. Coupled Flap-Lag Results*

The flap-lag problem has been first treated in Refs. 31 and 32. However, the best treatment of the problem is due to Ormiston and Hodges,[14] who have clearly identified the role of elastic coupling on this instability using a rigid, centrally hinged, spring-restrained model of the hingeless blade. A similar treatment, without the effect of elastic coupling, using an elastic blade and a fully nonlinear treatment of the problem, was given in Refs. 15 and 16.

The mathematical treatment presented above, applies to this problem when one assumes that the blade is torsionally rigid, i.e., $\phi = 0$. When the elastic coupling terms $E_{c1}$ and $E_{c2}$ are not included, one obtains ellipse-like stability boundaries as shown in Fig. 4, which was taken from Ref. 16. Combinations of rotating flap and lag frequencies, $\bar{\omega}_{F1}$ and $\bar{\omega}_{L1}$ inside the ellipse-like region, represent unstable blade configurations for values of collective pitch setting above the value of $\theta_c$ given on the curves (where $\theta_c$ is the critical collective pitch setting at which the linearized system becomes unstable).

The effect of elastic coupling on the stability boundaries is illustrated by Fig. 5 taken from Ref. 14. When the elastic coupling parameter $R = 0$, an ellipse-like region, similar to those in Fig. 4, is obtained. Increasing this parameter to $R = 0.5$ shifts the unstable region to very high values of $\bar{\omega}_{L1}$, which do not occur in practical blade configurations. This means basically that the unstable regions shown in Fig. 4 are eliminated by elastic coupling. Results presented in Ref. 33 also indicate that small amounts of viscous type of structural damping (~1% of critical) are sufficient to eliminate the unstable regions in Fig. 4.

In addition to the theoretical studies on the flap-lag type of instability, Ormiston and Bousman[34] have performed an experimental study that validated the theoretical results.[14] Furthermore, their findings indicated that in the stall regime an unexpected type of blade motion instability for torsionally rigid hingeless rotors was encountered. Use of quasisteady

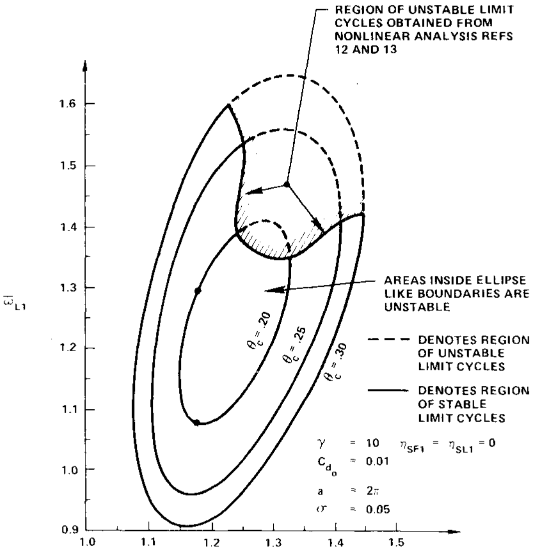


Fig. 4 Typical flap-lag stability boundary in hover without elastic coupling ($\sigma = 0.05$, $\gamma = 5$, $\eta_{SFI} = \eta_{SLI} = 0$; cross-hatched region indicates unstable limit cycles).

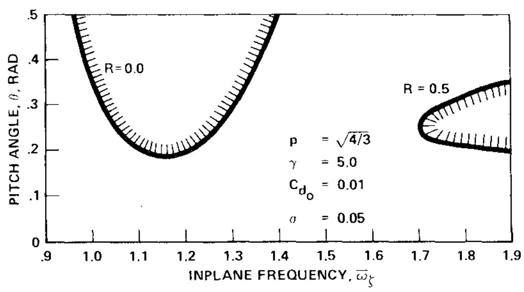


Fig. 5 Effect of elastic coupling on a typical flap-lag stability boundary (from Ref. 14, $R$ is the elastic coupling parameter, $p \equiv \bar{\omega}_{FI}$).

nonlinear airfoil section data incorporated in the linearized flap-lag theory was sufficient to give good correlation between theory and experiment. Elastic coupling is not successful in eliminating this type of instability, as indicated in Fig. 6 taken from Ref. 34.

It is interesting to note that similar conclusions have been obtained by Huber.[28] He concluded that at extremely high thrust conditions, a flap-lag instability caused by a reduction of flap damping due to aerodynamic stall is encountered. Elastic coupling effects have almost no effect on this instability.

From the description given above it is clear that the flap-lag type of instability is a result of destabilizing inertia and aerodynamic coupling effects. It is triggered by the lead-lag degree of freedom due to its low aerodynamic damping. Elastic coupling effects and small amounts of structural damping (less than 1%) are usually sufficient to eliminate this instability for most practical blade configuration in the linear (below stall) range of collective pitch setting.

*2. Coupled Flap-Lag-Torsion Results*

Preliminary studies on the coupled flap-lag-torsional stability of hingeless blades were presented in Refs. 15 and 33. In these studies, the elastic coupling effect was set equal to zero and the main purpose of these studies was to show the importance of the torsional degree of freedom.

Comprehensive results for a hingeless blade with distributed torsional properties were presented in Ref. 19. These results were based upon an analysis similar to the one outlined at the beginning of this section. The main conclusion

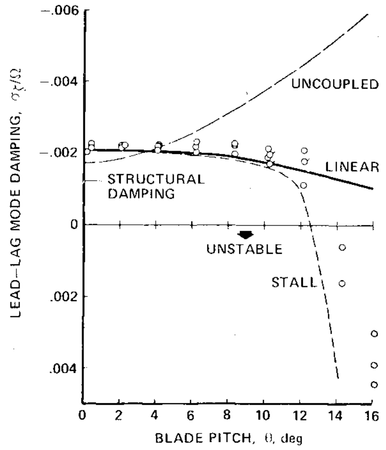


Fig. 6 Stall-induced flap-lag instability, dimensionless lead-lag damping (300 rpm, $R = 0.96$, $\bar{\omega}_\zeta \equiv \bar{\omega}_{LI} = 1.62$, $p \equiv \bar{\omega}_{FI} = 1.28$, from Ref. 34).

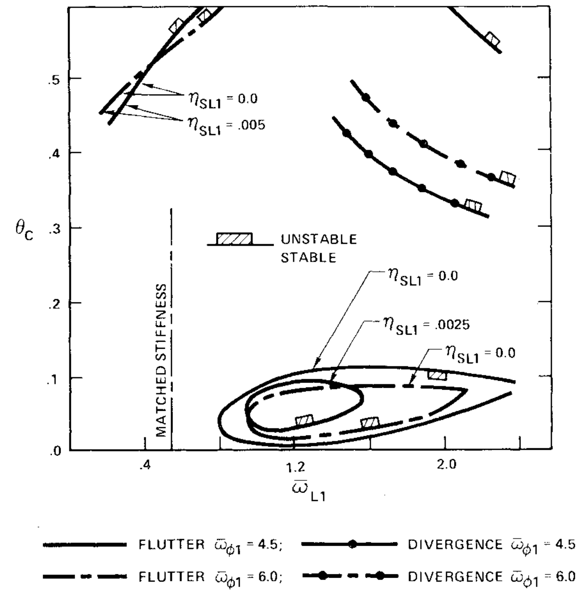


Fig. 7 Combined effect of precone and structural damping (for $\beta_p = 3$ deg, and $\bar{x}_A = 0$; $\sigma = 0.08$; $\gamma = 8.0$, $\bar{k}_0 = 0.02$, $\bar{\omega}_{FI} = 1.14$).

of this study was that, without precone, all practical blade configurations were found to be stable. With precone and relatively low values of torsional stiffness, a flap-lag type of instability was found to occur at low collective pitch settings.

Partial results on various aspects of the coupled flap-lag-torsion problem were also presented in Refs. 28, 35, and 36. The most detailed were those presented by Huber.[28] It was found that droop and sweep (see Fig. 3) which introduce strong coupling effects for hingeless rotor configuration have a very strong influence on blade stability.

The effect of blade cross-sectional offsets (see Fig. 2), structural damping precone and blade modeling assumptions was considered in Ref. 21. In a recent study by Powers,[30] the effect of droop, number of modes used in the analysis, offsets, pretwist, and nonuniform mass distribution, together with some additional effects, was also studied in a systematic manner.

A typical coupled flap-lag-torsional stability boundary taken from Ref. 21 is shown in Fig. 7. The main item of interest in this figure is the bubble-like region of instability occurring for low values of collective pitch $\theta$. This instability occurs only in the presence of precone and is a flap-lag type of instability. This instability was also observed in Refs. 19 and 36. The unstable region decreases as the torsional stiffness $\bar{\omega}_{\phi I}$

is increased from 4.5 to 6.0. The most important item, however, is the sensitivity of this unstable region to small amounts of viscous type of structural damping. For the case of low torsional frequency, $\bar{\omega}_{\phi l}=4.5$, .25% of critical damping considerably reduces this unstable region, while for $\bar{\omega}_{\phi l}=6.0$ the same amount of structural damping completely eliminates this instability, indicating that it is a weak one. Additional results[30] indicate that this region is quite sensitive to the number of modes used in the analysis. Thus in calculations that are intended to simulate experimental results, at least three uncoupled rotating modes for each elastic degree of freedom should be used in the analysis.

A considerable amount of additional results are presented by Powers.[30] The most important of these is that negative droop (undeformed elastic axis above the feathering axis) can induce a low collective pitch type of flap-lag instability similar to the one induced by precone, except that this instability cannot be eliminated by small amounts of structural damping.

Results showing the effects of variable structural coupling were recently published by Hodges and Ormiston.[24] These results indicate strong sensitivity of blade stability boundaries to structural coupling and precone which is similar to their previous conclusions.[19] Sensitivity of the stability boundaries to type and number of modes used in the analysis is also presented in Ref. 24.

In conclusion, coupled flap-lag-torsional analyses for hingeless blades indicate in general stable configurations. Various blade configurations can be destabilized by precone, droop, offsets, and negative built-in twist, which tends to reduce the stabilizing structural coupling effects. These parameters also can be used to tailor the aeroelastic behavior of the blade. Finally, it is interesting to note that the natural flutter parameter used in rotary-wing aeroelastic studies in hover is the collective pitch setting of the blade. This differs from the velocity that is used as the typical flutter parameter in fixed-wing aeroelastic problems.

### D. Effect of Unsteady Aerodynamics on the Coupled Flap-Lag-Torsional Problem in Hover

The coupled flap-lag and the coupled flap-lag-torsional problems presented in the previous section were treated with the assumption that the quasisteady approximation to the unsteady airloads is adequate. Furthermore, it was also assumed that the apparent mass terms except those representing damping in pitch are negligible. The basis for this assumption was the classical work of Miller and Ellis,[6] which indicated that in general the quasistatic solution appears to provide a reasonable, although slightly conservative, boundary for coupled flap-torsion flutter stability boundaries. At the same time, it is reasonably well known that under certain conditions unsteady wake effects can influence both the aeroelastic stability and the aeroelastic response of a rotor blade.[37] These conditions were found to occur primarily at low inflow or low collective pitch settings.[38]

A recent study by Anderson and Watts[39] has indicated that unsteady aerodynamics can significantly affect the aeroelastic stability of a hingeless rotor at relatively low collective pitch settings. In Ref. 39 only Loewy's[41] unsteady aerodynamic theory was used; however, the incorporation of the unsteady aerodynamic theory in the blade equations of motion was not carefully done. In particular, the unsteady aerodynamic coefficients as given in Ref. 39 are not consistent with a rotor blade having flap, lag, and torsional degrees of freedom, which is the case considered in Ref. 39. This is mainly due to the fact that while the blade has flap, lag, and torsional degrees of freedom the unsteady aerodynamic coefficients used in Ref. 39 are valid only for a blade having flap and torsional degrees of freedom. Lack of adequate documentation hinders any evaluation of the analytic aspects of this paper.

An interesting result taken from Ref. 39 is shown in Fig. 8. The figure shows the computed flutter boundaries for an

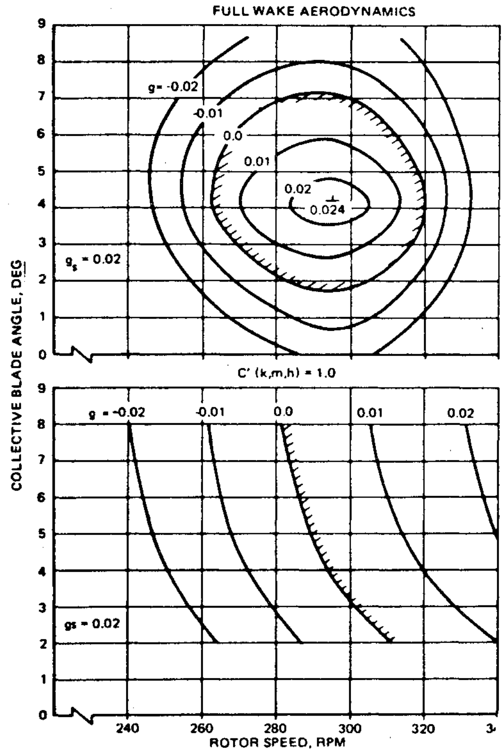


Fig. 8 Effect of wake on stability boundaries in hover (from Ref. 39, *g*-fictitious structural damping used in *V-g* method).

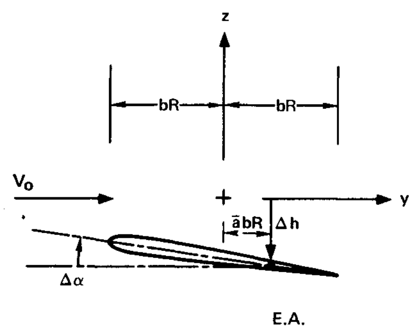


Fig. 9 Geometry of blade motion for conventional unsteady aerodynamics.

instability combined of a third flap, second torsion and second inplane bending mode, which was also experimentally encountered in whirl tower tests. The plot shows stability boundaries comparing full unsteady aerodynamics with those obtained when the quasisteady assumption, $C'(k,m,h)=1.0$, is made. For the unsteady aerodynamics case, including wake effects, the stability boundaries are circular in nature having a center at approximately 4.2 deg collective pitch and a rotor speed of 294 rpm. In contrast to the circular contours for the unsteady case, the quasisteady case contours as seen in the lower half of Fig. 8 are more nearly parallel. Qualitatively they exhibit significantly different behavior. The experimental evidence obtained in Ref. 39 correlated reasonably well with the unsteady prediction.

A systematic study of the effect of unsteady aerodynamics on the coupled flap-lag-torsional stability of hingeless helicopter blades has been recently presented by Friedmann and Yuan.[40,42] In this study, four different unsteady aerodynamic strip theories were modified and incorporated in a coupled flap-lag-torsional aeroelastic stability analysis for hover. The following theories were considered: 1) Theodorsen's incompressible fixed wing theory,[1] 2) Loewy's incompressible rotary wing theory,[41] 3) unsteady compressible fixed wing theory or Possio's theory,[1] and 4) unsteady two dimensional compressible rotary wing theory.[43,44] The geometry of the problem employed in the formulation of these theories is shown in Fig. 9.

The assumptions commonly used in deriving the various unsteady aerodynamic strip theories as illustrated by this figure are as follows: 1) Cross sections of the wing or blade are assumed to perform only simple harmonic pitching and plunging oscillations about a zero equilibrium position as indicated in Fig. 9. These are $\Delta \bar{h} = \Delta \bar{\bar{h}} e^{i\omega t}$, $\Delta \bar{\alpha} = \Delta \bar{\bar{\alpha}} e^{i\omega t}$. 2) The velocity of oncoming airflow is constant. 3) Usual potential small-disturbance unsteady aerodynamics are assumed to apply.

The basic difficulty encountered when attempting to apply the unsteady aerodynamic theories mentioned above to rotor blade aeroelastic calculations are primarily due to the fact that a rotor blade having flap, lag, and torsional degrees of freedom violates assumptions 1 and 2 given above. The main differences between the assumptions and the real behavior of a rotor blade are the following: 1) in addition to the constant velocity of oncoming flow, the blade also experiences a time-dependent velocity variation due to its lead-lag motion; 2) in addition to the harmonic variation in the angle of pitch, due to elastic torsional motion of the blade, a constant collective pitch setting is also imposed on the airfoil; 3) the plunging velocity of the airfoil $\Delta \dot{\bar{h}}$, is composed not only of a harmonic part associated with elastic flapping motion but, in addition, has a constant velocity component due to the inflow through the rotor disk; 4) blade deflections are not necessarily small when compared to the thickness of the airfoil.

In Refs. 40 and 42 the four unsteady strip theories listed previously were modified in two stages. First, terms accounting for the variations in oncoming velocity and constant angle of pitch and constant inflow were introduced. Next the airloads were modified to make them compatible with the deformation field of a rotor blade having flap, lag, and torsional degrees of freedom.

The modified airloads were incorporated in an aeroelastic analysis similar to the one described in the previous section and the sensitivity of the coupled flap-lag-torsional aeroelastic stability boundaries was investigated. It should be noted that determination of the stability boundaries for this case is much more complicated than the simple eigenanalysis described in the previous section; a suitable algorithm is given in Refs. 40 and 42.

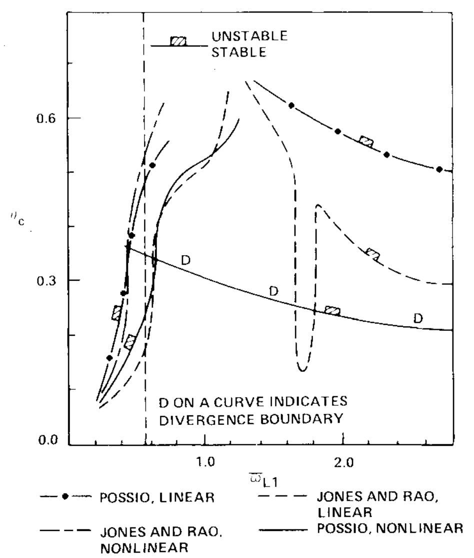


**Fig. 10 Effect of modified rotary-wing compressible aerodynamics on a typical case with two modes in each elastic degree of freedom ($D$ on a curve indicates divergence boundary, $\gamma = 8.0$, $\sigma = 0.08$, $b = 0.0313$, $\beta_p = \eta_{SF1} = \eta_{SL1} = 0$, $k_0 = \bar{k}_A = 0.02$, $\bar{\omega}_{F1} = 1.14209$; $\bar{\omega}_{T1} = 6.17018$; $\bar{x}_A = 0.20$, $M = 0.50$ and linear or nonlinear static equilibrium).**

It was found that for some cases apparent mass terms and, in particular, compressibility should be included. Quasisteady aerodynamics, with apparent mass terms neglected, tended to yield conservative boundaries. Under certain conditions, when two modes were used to represent each elastic flap, lag, and torsional degree of freedom, wake effects could be important, particularly when an offset between the aerodynamic center and the elastic axis is present in the system.

Typical results[40,42] are shown in Fig. 10. The left branch of the stability boundary is the flap-lag branch which is essentially unaffected by the returning wake and the unsteady aerodynamics; however, it is sensitive to the number of modes used and the static equilibrium position about which the equations are linearized. Two rotating uncoupled modes are used for each elastic degree of freedom. The flap-lag branch exhibits considerable sensitivity to the incorporation of the nonlinear equilibrium position. The right-hand part of the stability boundary represents coupled flap-lag-torsional oscillations, in which the role of torsion is dominant. The most interesting result in Fig. 10 is the narrow unstable region, occurring with Jones' and Rao's theory, which dips slightly below the divergence boundary. These regions[40,42] were obtained only with modified unsteady theories containing the effect of the returning wake and with a multimode-type analysis. From a qualitative point of view, these regions resemble a type of instability similar to those presented on the top of Fig. 8. In this region, the second flap mode appears to interact with the first torsional mode. Somewhat similar results were also obtained[42] with Loewy's modified theory.

## III. Rotary-Wing Aeroelastic Problems in Forward Flight

### A. Introduction and Review of Methods Available for Treating the Periodic Coefficient Problem

The forward flight condition introduces considerable complications to rotary-wing aeroelastic stability and response problems. From the aerodynamic point of view, it leads to a much more complicated representation of the unsteady aerodynamic forces. Furthermore, it results in a region of reversed flow, which is also accompanied by locally stalled flow in the retreating blade region. The computation of the unsteady aerodynamic loads on a rotor blade in forward flight is a formidable task in computational fluid mechanics.

The aeroelastic problem in forward flight is further complicated by the appearance of periodic coefficients in the equations of motion. In the past, there has been a preoccupation with equations with periodic coefficients in rotary-wing dynamics and aeroelasticity; a brief review of this effort can be found in Refs. 45 and 46.

When the unsteady aerodynamic problem posed by forward flight is disregarded and the unsteady aerodynamic loads are evaluated using blade element theory as has been done in many aeroelastic studies, the treatment of the coupled flap-lag or coupled flap-lag-torsional problem in forward flight is reduced to the derivation of an algebraically complicated set of equations of motion with periodic coefficients.

A number of methods are available for dealing with the stability of these systems. One method deals with the equations when they are written in a blade-fixed rotating coordinate system. For this case, it can be shown[45] that the most convenient method for dealing with the *linearized* aeroelastic problem is to use multivariable Floquet-Liapunov theory to determine the stability boundaries of the system.[45,46] Multivariable Floquet-Liapunov theory was introduced into rotary-wing aeroelasticity by Hall,[47] who considered the coupled flap-lag problem in a somewhat inconclusive manner, and by Peters and Hohenemser,[48] who applied it to the flapping problem of lifting rotors in forward flight. It should be emphasized that this method is general and valid for *arbitrary* advance ratios.

Another method is suitable for the case when the stability of the system at relatively low advance ratios, $\mu \leq 0.4$, is required. For this case the blade-fixed generalized coordinates can be transformed into a nonrotating hub-fixed generalized coordinates, usually called multiblade coordinates or quasinormal coordinates.[9,49] Application of this transformation to the original system yields a transformed set of equations where some of the periodic terms are transformed into constant coefficient terms. The remaining periodic terms in the multiblade equations contain only third harmonics for a three-bladed rotor[50] and second and fourth harmonics for a four-bladed rotor.[49] Omitting these periodic terms yields a constant coefficient approximation to the periodic system that is acceptable below $\mu \leq 0.3$ in most cases. It should be noted that when the periodic terms are not neglected, Floquet-Liapunov theory has to be used in the solution.

This method has been used by Hohenemser and Yin[49,51] to study the flapping and flap-torsional stability and response of rotors, and by Biggers[50] to study the flapping stability of a helicopter rotor in forward flight. More recently, Kaza and Hammond[52] have applied a slightly modified version of this method to the flap-lag problem in forward flight and found the method to be satisfactory for advance ratios $\mu \leq 0.4$.

The third method for dealing with the problem of equations with periodic coefficients in their complete nonlinear form is based upon direct numerical integration either in the blade-fixed or nonrotating coordinate system. In this case, time histories of blade motions or blade dynamic loads or stresses are obtained, from which the stability or response of the system can be determined. This method is widely used in industry.[23,53] While for subcritical response calculations this method is probably adequate, it might be unreliable for stability calculations, due to the mathematical properties of periodic nonlinear systems. For this reason, it might be desirable to augment direct numerical integration by an eigenanalysis type of calculation such as that presented in the following section.

### B. Coupled Flap-Lag and Coupled Flap-Lag-Torsional Aeroelastic Problems in Forward Flight

A number of studies dealing with the effect of forward flight on the coupled flap-lag stability problem have been conducted in the past.[15,16,31,32,45-47,54,55] While the older studies have been instrumental in gaining, slowly, an understanding of the problem, it is only the more recent work presented in Refs. 46, 54, and 55 that presents a consistent treatment of this problem.

The coupled flap-lag-torsional problem in forward flight has received only limited attention in the rotary-wing literature. The coupled, linearized, flap-lag-torsion motion has been investigated by Crimi[56] using a modified Hill method; only a limited number of somewhat inconclusive numerical results were presented. A detailed, comprehensive, and consistent treatment of the coupled-flap-lag-torsional problem of a bearingless hingeless rotor in forward flight has been recently completed by Bielawa.[23,27] In this study, nonlinear equations were derived and the proper trim conditions were incorporated in the aeroelastic problem. Time histories of the blade response were obtained by direct numerical integration.

Another comprehensive treatment of the coupled-flap-lag-torsional problem in forward flight is presented in a recently completed study.[67] In this study nonlinear equations were obtained that were consistently linearized about a time-dependent equilibrium position determined from the trim conditions. The stability boundaries were obtained by performing an eigenanalysis based upon multivariable Floquet-Liapunov theory.

Furthermore, a number of helicopter companies have computer programs capable of dealing with the coupled rotor fuselage aeroelastic problem in forward flight. Obviously the coupled flap-lag or coupled flap-lag-torsional aeroelastic problem is simply a component, or subset, of this more complex problem. These problems will be reviewed in the next section.

The salient features of the rotary-wing aeroelastic problem in forward flight can be best illustrated by extending the symbolic outline of the analysis presented for hover, in Sec. IIB, to the case of forward flight. For convenience, only the flap-lag problem will be considered, following Ref. 46; the extension to the case of coupled flap-lag-torsion is straightforward.[26] The geometry of the problem is again represented by Figs. 1 and 2, except that for this case $\phi = 0$. The basic equations are Eqs. (1) and (2), and the torsional degree of freedom is suppressed. The ordering scheme used is similar to the previous case, except that one also introduces[46]

$$\mu = O(1) \qquad \theta_{1s} = O(\epsilon_D) \qquad \theta_{1c} = O(\epsilon_D) \tag{20}$$

The displacement field is given again by Eqs. (8-10), and Galerkin's method is used to eliminate the spatial dependence of the problem using the modes given by Eqs. (11).

A significant difference between hover and forward flight is due to time varying pitch

$$\theta_G = \theta_0 + \theta_t = \theta_0 + \theta_{1s}\cos\psi + \theta_{1s}\sin\psi \tag{21}$$

which, using Eqs. (5-7), introduces a time-varying elastic coupling, and also a time-varying torsion-flap-lag coupling when $\phi \neq 0$.

The various inertia and aerodynamic loads, including the effects of reversed flow, corresponding to this problem can be found in Ref. 46. Another important difference between hover and forward is due to the fact that the cyclic pitch components $\theta_{1s}$, $\theta_{1c}$ and the inflow $\lambda$ have to be determined from the trim condition of the rotor, which is shown schematically in Fig. 11.

The trim conditions can be calculated[46] using two separate trim procedures.

*1. Propulsive Trim*

In this trim procedure, which simulates actual forward flight conditions, the rotor is maintained at a fixed value of the thrust coefficient $C_T$ with forward flight. Horizontal and vertical force equilibrium is maintained as well as zero pitching and rolling moments.

*2. Wind-Tunnel Trim or Moment Trim*

In this trim procedure, which simulates conditions under which the rotor would be tested in a wind tunnel, pitching and rolling moments on the rotor are maintained at zero. Horizontal and vertical force equilibrium is not required for this case, because the rotor is mounted on a supporting structure.

Due to these trim requirements, the equilibrium position about which the nonlinear equations of motion have to be linearized is a *time-dependent one*. This time-dependent

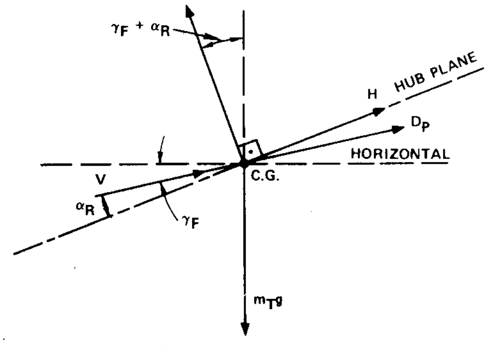


**Fig. 11 Geometry for trim calculation.**

equilibrium position is a result of the cyclic pitch variation required to trim the rotor in forward flight. Assuming that first harmonic terms are sufficient for the representation of the time-dependent equilibrium position one has

$$\bar{g}_I(\psi) = g_I^0 + g_{Ic}\cos\psi + g_{Is}\sin\psi \tag{22}$$

$$\bar{h}_I(\psi) = h_I^0 + h_{Ic}\cos\psi + h_{Is}\sin\psi \tag{23}$$

Substitution of Eqs. (22) and (23) into the differential equations results in a system of six algebraic equations from which the equilibrium position is determined.

$$[S]\{q_0\} = \{C\} + \{f_{NL}(g_I^0, g_{Is}, g_{Ic}, h_I^0, h_{Is}, h_{Ic})\} \tag{24}$$

where $\{q_0\}^T = \lfloor g_I^0, g_{Ic}, g_{Is}, h_I^0, h_{Ic}, h_{Is} \rfloor$ and the elements of the matrices $[S]$ and $\{C\}$ are given in Ref. 46. The vector function $\{f_{NL}\}$ contains nonlinear combinations of the type $g_I^0\ h_{Is}$, $g_I^0\ h_{Ic}$...etc. In Ref. 46, these terms were neglected; recent results, based on the full solution of Eqs. (24), indicate[55] that under certain conditions these terms can affect the stability boundaries.

The process of linearization of the equations of motion consist of expressing the time dependence of the generalized coordinates of the elastic displacement field as the sum of a time-dependent equilibrium value about which time-dependent perturbations occur:

$$g_I(\psi) = \bar{g}_I(\psi) + \Delta g_I(\psi) \tag{25a}$$

$$h_I(\psi) = \bar{h}_I(\psi) + \Delta h_I(\psi) \tag{25b}$$

When Eqs. (25) are substituted into the nonlinear ordinary differential equations of motion, nonlinear terms are transformed into coupling terms and terms containing squares of the perturbation quantities $\Delta g_I$ and $\Delta h_I$ are neglected. Finally, the equations are transformed into state variable form by introducing

$$\Delta g_I' = y_1 \quad \Delta h_I' = y_3 \quad \Delta g_I = y_2 \quad \Delta h_I = y_4 \tag{26}$$

The equations of motion in their final form are

$$\{y'\} = [A(\psi)]\{y\} \tag{27}$$

Comparing Eqs. (16) and (26), it is evident that both aeroelastic problems are quite similar except for forward flight $[A(\psi)]$ is a periodic matrix. Using multivariable Floquet-Liapunov theory[45] the stability investigation of blade motions is straightforward. Based on the Floquet-Liapunov theorem, the transition matrix for a periodic system, having a common period $T$, can be written as

$$[\Phi(\psi,\psi_0)] = [P(\psi)]^{-1} e^{[R](\psi-\psi_0)} [P(\psi_0)] \tag{28}$$

where $[P(\psi)]$ is also a periodic matrix and $[R]$ is a constant matrix related to the value of the transition matrix at the end of a period

$$[\Phi(T,0)] = e^{[R]T} \tag{29}$$

The stability of the system is governed by the characteristic exponents or the eigenvalues of $[R]$ denoted by

$$\bar{\lambda}_k = \bar{\zeta}_k + i\bar{\omega}_k \tag{30}$$

The system is stable when

$$\bar{\zeta}_k < 0 \qquad (k = 1,2,\ldots n)$$

Comparing Eqs. (19) and (30), it is clear that the real part of the characteristic exponent for the periodic system $\bar{\zeta}_k$ plays a role similar to that of modal damping $\zeta_k$ in the constant coefficient system. The key to the efficient numerical treatment of periodic systems is the numerical computation of the transition matrix at the end of one period $[\Phi(T,0)]$.

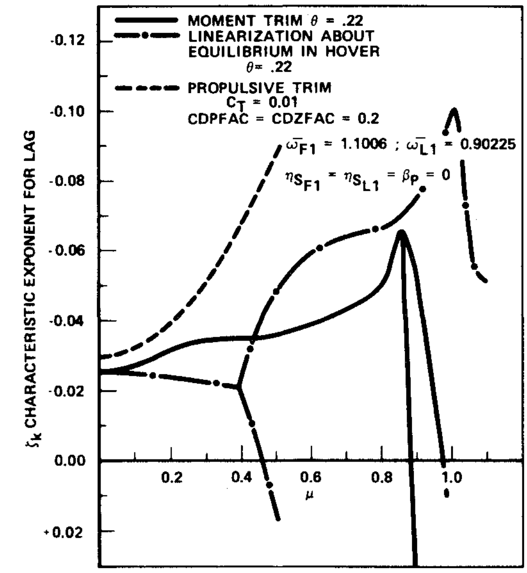


**Fig. 12 Effect of propulsive trim and moment trim on a soft in-plane blade ($\gamma = 10$, $\sigma = 0.01$, $C_{DP} = 0.01$, $b = 0.0313$).**

Two efficient numerical schemes are available for performing this task. One is a generalization of the rectangular ripple method[45] and the other is an improved numerical integration scheme described in Ref. 52. Both represent essentially a single-pass integration for obtaining the transition matrix at the end of one period resulting approximately in an $n$-fold saving of computing time for an $n$th order $[A(\psi)]$ matrix when compared to previous methods.[48] A detailed description of the numerical details of these methods is presented in Ref. 57.

Typical results[46] for the coupled flap-lag problem in forward flight are shown in Fig. 12. To illustrate the inherently nonlinear nature of the problem the equations were linearized about three different equilibrium positions: an artificial static equilibrium position defined by $g_I^0$ and $h_I^0$, one obtained from moment trim, and one obtained from propulsive trim. As shown, the critical advance ratio $\mu$ at which the lead-lag degree of freedom becomes unstable is quite dependent upon the equilibrium position about which the equations are linearized. Furthermore, the blade can become unstable at realistic values of the advance ratio $\mu$.

The effect of the torsional degree of freedom on the coupled flap-lag problem is illustrated by Fig. 13 taken from Ref. 67. For $\bar{\omega}_{T_I} = 60$, the torsional degree of freedom is locked out; at $\bar{\omega}_{T_I} = 15.03$ torsion is almost suppressed while with $\bar{\omega}_{TI} = 3.0$ one has a relatively soft blade in torsion. Clearly introduction of torsion for this case significantly destabilizes the in-plane degree of freedom. Additional results indicate that below $\bar{\omega}_{TI} < 9.0$ the torsional degree of freedom should be included for a realistic representation of blade dynamics.

In addition to the aeroelastic stability problem in forward flight, one of the major topics of rotary-wing aeroelasticity is the aeroelastic response problem or dynamic loads problem in forward flight. This problem represents a combination of unsteady aerodynamics and structural dynamics. This area has been recently reviewed by Dat[8,58] with an emphasis on unsteady aerodynamics. A comprehensive review of the state-of-the-art for predicting dynamic loads on helicopter rotor has been prepared recently by Ormiston.[59]

These reviews indicate that a considerable amount of theoretical work, primarily in the modeling of unsteady

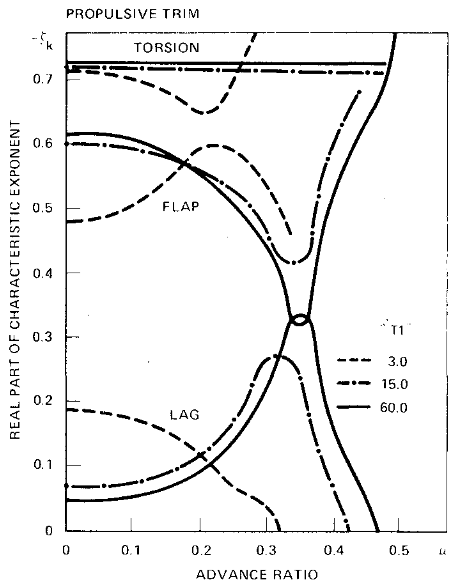


Fig. 13 Real part of characteristic exponents for flap, lag, and torsion vs advance ratio ($\bar{\omega}_{F1} = 1.082$, $\bar{\omega}_{L1} = 1.4171$, $\gamma = 5.0$, $C_w = 0.01$, $C_{DP} = 0.01$).

aerodynamic wake effects, has to be done before the dynamic fully aeroelastic loads on a rotor blade in forward flight, at relatively low advance ratios, can be accurately predicted.

## IV. Complete Rotor and Coupled Rotor/Fuselage Aeroelastic Problems

The rotary-wing aeroelastic problems described in the previous sections were restricted basically to single-blade or isolated-blade aeroelastic problems. In reality, interblade mechanical coupling or coupling between rotor and the fuselage or coupling between the rotor/fuselage and the control system can have a significant effect on the aeroelastic stability and response of this complex aeroelastic system. A number of these problems pertaining to hingeless rotor flight dynamics have been reviewed in great detail by Hohenemser.[9]

During the last few years a number of complex analyses dealing with the coupled rotor/fuselage problem have been developed by the helicopter industry and have been implemented by sophisticated computer programs. Results from these programs have been compared to flight test and wind-tunnel test results. These comparisons have usually indicated good qualitative predictive capability; however, the quantitative predictive capability was in some cases less than satisfactory. Thus these programs have become valuable tools in the process of designing rotor systems with favorable aeroelastic characteristics.

In many cases, however the mathematical details, basic assumptions, and detailed documentation are not available, and for this reason they cannot be reviewed in detail.

One such program, which is not adequately documented, has been originally developed by Messerchmitt-Bolkow-Blohm for the study of the BO-105 air resonance problem; it has also been adopted and modified by Vertol. This program has been used extensively to generate results that have been used subsequently in correlation studies with both dynamic model tests and with extensive full-scale tests.[9,28,36,60,61]

A schematic representation of the coupled rotor fuselage model for this program is shown in Fig. 14. In this model, the elastic cantilevered blade is represented by a spring-restrained, hinged, rigid blade. Three hinges are used to simulate the first flap, first lag, and first torsion modes, in that order, from inboard to outboard. In addition, a pitch degree of freedom is provided inboard of the flap hinge to facilitate the simulation of any torsional stiffness distribution relative to the flap and lag hinges. The blade model includes precone, blade sweep,

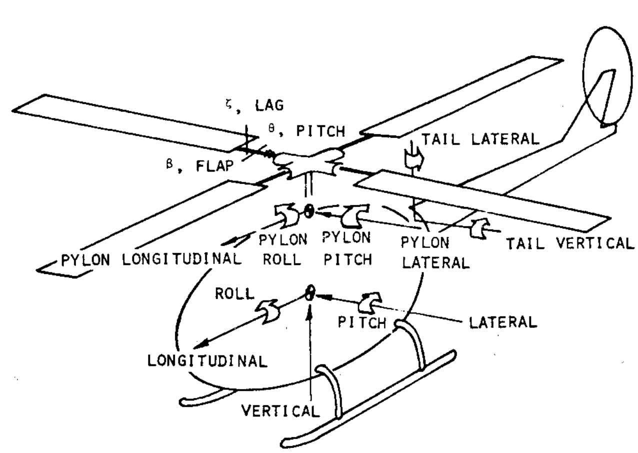


Fig. 14 Coupled rotor-fuselage analytic model (C-56 program, taken from Ref. 61).

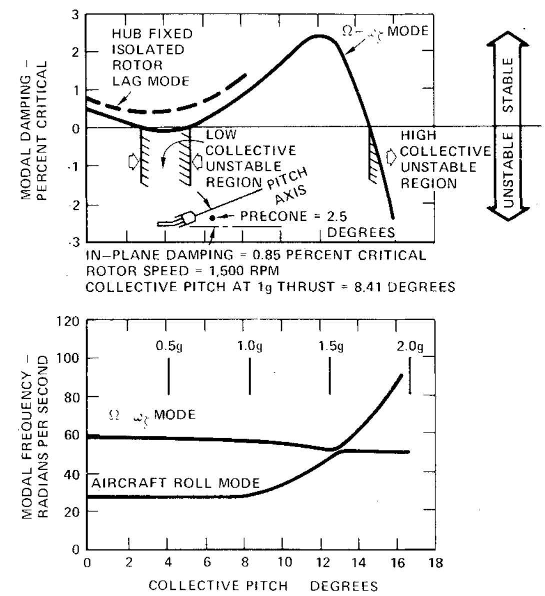


Fig. 15 Effect of collective pitch on air resonance characteristics (taken from Ref. 36).

kinematic pitch flap and pitch lag coupling, and a variable chordwise center of gravity.

The aerodynamic model is based on blade element theory and can handle hover, forward flight, and maneuver flight conditions. It uses two-dimensional airfoil data with stall, reversed flow, and compressibility effects.

Quasinormal or multiblade coordinates are used to transform the blade equations of motion into a nonrotating system. The airframe has five rigid-body degrees of freedom (longitudinal, lateral, vertical, pitch, and roll) and two flexible degrees of freedom (pylon pitch and roll).

The equations of motion are nonlinear and are solved by a numerical time-history solution technique. Subsequent plotting of the time history of each degree of freedom is used to obtain frequencies, amplitudes, phases, and damping coefficients.

One of the main purposes of this program was the simulation of air resonance aeroelastic stability boundaries of a soft-inplane hingeless-rotor helicopter.[28,36,60] The air resonance mode is usually the $\Omega - \omega_\zeta$ lag mode (where $\omega_\zeta$ is the inplane fundamental frequency), which can potentially combine with the aircraft pitch and/or roll mode. However, aircraft fuselage pitch and roll modes are usually heavily damped due to flap motion and therefore stable.

An appreciation of the coupled rotor fuselage aeroelastic problem can be obtained from Fig. 15, which was taken from

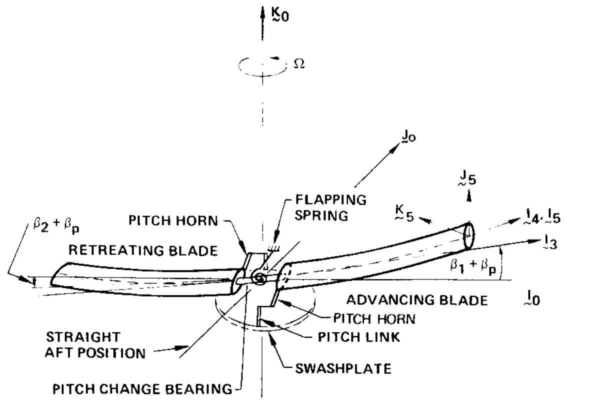


**Fig. 16 Deformed configuration, used in analysis of complete two-bladed teetering rotors.**

Ref. 36. Examination of Fig. 15 indicates that the low collective pitch type of instability, which is a pure flap-lag type instability, has little coupling with the roll mode; however, the high collective pitch case has significant coupling between the regressive lag $\Omega-\omega_\zeta$ mode and the aircraft roll mode.

Another interesting item shown in the figure deals with the effect of fuselage motion on the low collective flap-lag mode in the presence of precone, shown previously in Fig. 7. Comparing the damping levels in this mode when the hub is fixed with the damping levels when the fuselage degrees of freedom are included clearly shows that the fuselage degrees of freedom can significantly destabilize this mode. It should be noted that the low collective precone-induced mode manifests itself usually as an oscillation at the lag frequency which contains predominant lag motion. For this reason it was identified in Ref. 36 as a "pure lag mode." This example clearly illustrates that an understanding of single-blade aeroelastic problems is an important ingredient in understanding the coupled rotor/fuselage aeroelastic problem and vice versa.

The reasonably good correlation between model, flight test, and digital simulation obtained with this program, indicates that its detailed documentation, so that it could become available in the public domain, would be in the best interests of the technical community.

Another general-purpose coupled rotor/fuselage analysis has been developed by Johnston and Cassarino.[22] This program which contains well-documented equations has considerable capability for simulating a variety of coupled rotor/fuselage aeroelastic problems for a variety of blade configurations. Published results[62] indicate reasonable correlation with test results.

A third computer program with similar capabilities is the REXOR program which has been developed by Lockheed. The mathematical rotorcraft simulation technique, mathematical model, and correlation between simulated and test results are described in Ref. 35. Due to the fact that the rotor is gyro controlled, this program represents an advanced complete rotor/fuselage/control system type of aeroelastic treatment. This rotorcraft simulation program has been recently documented in a relatively complete manner.[68] Two minor deficiencies in the documentation of this impressive computer program are 1) lack of a clear statement of the assumptions used in deriving the blade equations of motion and 2) lack of a clear description of the treatment of the nonlinear terms in the equations. This is a considerable shortcoming because the simulation is based upon the direct time history solution of the nonlinear equations.

Another analysis and computer program implementation intended for coupled rotor fuselage *vibration studies* at high-speed flight has been presented by Gerstenberger and Wood.[65] While individual blade flap, lag, and fuselage degrees of freedom are coupled and proper trim conditions are used, the analysis is limited by the assumption that blades are torsionally rigid.

A review of the coupled rotor fuselage aeroelastic problems would be incomplete without briefly mentioning the aeroelastic problems associated with tail rotors. These problems have been treated with considerable detail in Refs. 63 and 64, which consider two-bladed teetering and three- and four-bladed gimbaled tail rotors. Unlike a main rotor, a tail rotor is not trimmed for wind or flight velocities with cyclic pitch. It operates in extremely adverse aerodynamic and dynamic environment and must produce both positive and negative thrust. Interference between main rotor and tail rotor leads to an unsteady aerodynamics problem of exquisite intractability.

According to Ref. 64, the main tail rotor aeroelastic problems encountered are as follows: 1) Tail wagging, which consists of tail rotor blade flapping coupled with tail boom second lateral bending-torsion mode. Tail rotor drive shaft frequency, when close to tail boom frequency, can introduce torque changes, resulting in reduced aerodynamic damping and leading to further amplification of this instability. This instability is eliminated by introducing appropriate pitch-flap coupling. 2) Blade motion instability of flap-lag type at high advance ratios for $\bar{\omega}_{L1} \cong 1.5$. This can be eliminated by reducing the Lock number and increasing $\bar{\omega}_{L1} > 2.0$.

A more modest complete rotor aeroelastic problem, where interblade structural and mechanical coupling is clearly important, is the teetering rotor aeroelastic problem[54,55] shown in Fig. 16. It was shown, using consistently linearized equations, that the complete rotor stability boundaries and damping levels are quite different from those obtained when a single-blade type analysis, based on the assumption that no root moment is transferred from one blade to another, is performed.

## V. Concluding Remarks

In this survey of recent research on rotary-wing aeroelasticity, an attempt was made to emphasize the inherent nonlinear nature of the rotary-wing aeroelasticity when compared to fixed-wing aeroelasticity. The nonlinearities which can be due to both moderately large deflections and nonlinear aerodynamic effects can be important for both aeroelastic stability and response calculations. However, they probably have a stronger effect on stability than on response calculations. Thus, care and consistency in the formulation of rotary-wing aeroelastic analyses and mathematical models is of crucial importance. The papers and topics reviewed were rather arbitrarily selected; nevertheless, the material surveyed gives an indication regarding future trends in rotary-wing aeroelasticity. It is apparent that problems, formulations, and methods of solution are becoming more sophisticated and computerized. The realistic rotary-wing aeroelastic problem is obviously the complete coupled rotor-fuselage-control system aeroelastic problem. Satisfactory solutions to this problem will become available only after some intermediate problems are adequately solved. First, a reliable experimental data base of model and full-scale aeroelastic test results should be developed against which analyses such as coupled flap-lag-torsional analyses in forward flight can be validated. A similar approach regarding the coupled rotor-fuselage and dynamic load and response calculations should be taken. Additional research on unsteady aerodynamics around realistic rotor configurations in forward flight should be initiated. These theories should be developed with the aeroelastician as the potential user in mind; otherwise these theories might not be suitable for aeroelastic applications. Finally, comparisons between predicted and experimental results should be based on modern system identification methods.

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